Technical Note 1966-31

E. M. Hofstetter

An Efficient Technique for the Calculation of Velocity-Acceleration Periodograms

18 May 1966

Prepared for the Advanced Research Projects Agency
under Electronic Systems Division Contract AF 19(628)-5167 by

Lincoln Laboratory
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Lexington, Massachusetts

The work reported in this document was performed at Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology. This research is a part of Project DEFENDER, which is sponsored by the U.S. Advanced Research Projects Agency of the Department of Defense; it is supported by ARPA under Air Force Contract AF 19(628)-5167 (ARPA Order 498).

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LINCOLN LABORATORY

AN EFFICIENT TECHNIQUE FOR THE CALCULATION OF VELOCITY-ACCELERATION PERIODOGRAMS

E. M. HOFSTETTER

Group 42

TECHNICAL NOTE 1966-31

18 MAY 1966

LEXINGTON MASSACHUSETTS

ABSTRACT

The Cooley-Tukey method for greatly reducing the number of computations required to evaluate a velocity periodogram has been extended to the evaluation of velocity-acceleration periodograms. For N data points, this method requires approximately a factor of 2/3 N fewer computations than would be required by straightforward evaluation of the periodogram.

Accepted for the Air Force
Franklin C. Hudson
Chief, Lincoln Laboratory Office

An Efficient Technique for the Calculation
of Velocity – Acceleration Periodograms

The velocity – acceleration periodogram associated with the (complex) data samples r_o, ... r_{N-1} is defined by

$$P(f,\alpha) = \sum_{k=0}^{N-1} r_k \, e^{j2\pi fk\Delta} \, e^{j2\pi\alpha(k\Delta)^2} \tag{1}$$

where Δ denotes the (uniform) time separation between successive data points. P is periodic in f with period Δ^{-1} and periodic in α with period Δ^{-2} so that P need only be evaluated over the (f,α) region defined by $0 \leq f < \Delta^{-1}$, $0 \leq \alpha < \Delta^{-2}$. Furthermore, since the velocity and acceleration resolutions of the periodogram are given (approximately) by $(N\Delta)^{-1}$ and $(N\Delta)^{-2}$ respectively, it is usually sufficient to evaluate P at the discrete points given by $f = n(N\Delta)^{-1}$, $\alpha = m(N\Delta)^{-2}$ where n = 0, 1, ... N-1 and m = 0, ... N^2-1. These considerations transform the original periodogram problem to the evaluation of the expression:

$$P(n,m) = \sum_{k=0}^{N-1} r_k \, W^{nk} \, V^{mk^2} \tag{2}$$

where n = 0, 1, ... N-1; m = 0, 1, ... N^2-1; $W = \exp(j\ 2\pi/N)$, $V = \exp(j\ 2\pi/N^2)$.

Following Cooley and Tukey*, we assume that $N = 2^p$ and proceed to express the integers k, n, m in binary form as follows:

$$k = k_{p-1}2^{p-1} + \ldots + k_1 2 + k_o$$

$$n = n_{p-1}2^{p-1} + \ldots + n_1 2 + n_o$$

$$m = m_{2p-1}2^{2p-1} + \ldots + m_1 2 + m_o$$

where k_i, n_i and m_i take on the values 0 and 1. In addition, it will be convenient to express k^2 in the form

$$k^2 = (k^2)_{p-1} + \ldots + (k^2)_o$$

where $(k^2)_{p-\ell} \equiv$ those terms in k^2 that depend on $k_{p-\ell}$ but <u>not</u> on $k_{p-\ell+1}, \ldots k_{p-1}$. Thus,

$$(k^2)_{p-\ell} = k_{p-\ell}\, 2^{p-\ell+1} \sum_{g=\ell+1}^{p} k_{p-g}\, 2^{p-g} + k^2_{p-\ell}\, 2^{2(p-\ell)}$$

The derivation of this last formula is straightforward exercise. Note that

---

* Cooley and Tukey, An Algorithm for the Machine Calculation of Complex Fourier Series, Math. of Comp. <u>19</u>; April, 1965.

$(k^2)_{p-\ell}$ contains a factor $2^{p-\ell+1}$ except when $\ell = p$.

Next we note that

$$W^{nk} = W^{(n_o + \dots n_{p-1} 2^{p-1})(k_o + \dots + k_{p-1} 2^{p-1})}$$

$$= W^{k_{p-1} 2^{p-1} n_o} W^{k_{p-2} 2^{p-2}(n_o + n_1 2)}$$

$$\dots W^{k_o(n_o + \dots + n_{p-1} 2^{p-1})}$$

and

$$V^{m(k^2)} = V^{(m_o + \dots + m_{2p-1} 2^{2p-1})[(k^2)_o + \dots + (k^2)_{p-1}]}$$

$$= V^{(k^2)_{p-1}(m_o + \dots + m_{p-1} 2^{p-1})} \dots$$

$$V^{(k^2)_1(m_o + \dots + m_{2p-3} 2^{2p-3})} V^{(k^2)_o(m_o + \dots + m_{2p-1} 2^{2p-1})}$$

because the exponent of W need only be computed modulo $N = 2^p$ and the exponent of V need only be computer modulo $N^2 = 2^{2p}$.

With some obvious changes of notation, equation (2) now can be written in the form

$$P(n_o, \dots n_{p-1}, m_o, \dots m_{2p-1}) =$$

$$= \sum_{k_o} W^{k_o(n_o + \dots + n_{p-1} 2^{p-1})} V^{(k^2)_o(m_o + \dots + m_{2p-1} 2^{2p-1})}$$

$$\dots \sum_{k_{p-1}} r(k_o, \dots k_{p-1}) \; W^{k_{p-1} 2^{p-1} n_o} \; V^{(k^2)_{p-1}(m_o + \dots + m_{p-1} 2^{p-1})} \qquad (3)$$

For computational purposes, it is convenient to think of equation (3) as a sequence of p calculations as follows: First compute

$$P_1(k_o, \dots k_{p-2}, n_o, m_o, \dots m_{p-1})$$

$$\equiv \sum_{k_{p-1}} r(k_o, \dots k_{p-1}) \; W^{k_{p-1} 2^{p-1} n_o} \; V^{(k^2)_{p-1}(m_o + \dots m_{p-1} 2^{p-1})} \qquad (4)$$

then successively compute P_ℓ from $P_{\ell-1}$, $\ell = \ell, \dots p-1$, according to the formula

$$P_\ell(k_o, \dots k_{p-\ell-1}, n_o, \dots n_{\ell-1}, m_o, \dots m_{p+\ell-2}) =$$

$$= \sum_{k_{p-\ell}} P_{\ell-1}(k_o, \dots k_{p-\ell}, n_o, \dots n_{\ell-2}, m_o, \dots m_{p+\ell-3})$$

$$W^{k_{p-\ell} 2^{p-\ell}(n_o + \dots n_{\ell-1} 2^{\ell-1})} \; V^{(k^2)_{p-\ell} (m_o + \dots + m_{p+\ell-2} 2^{p+\ell-2}} \qquad (5)$$

Finally, P_p is computed from the formula,

$$P_p(n_o, \dots n_{p-1}, m_o, \dots m_{2p-1})$$

$$= \sum_{k_o} P_{p-1}(k_o, n_o, \dots n_{p-2}, m_o, \dots m_{2p-3})$$

$$W^{k_o(n_o + \dots n_{p-1} 2^{p-1})} \; V^{(k^2)_o(m_o + \dots + m_{2p-1} 2^{2p-1})} \qquad (6)$$

The last computed function P_p is the desired function P given by equation (3).

A straightforward computation of the periodogram from equation (2) would require $(N-1)N^3$ computations. (A computation is defined as being the performance of two complex multiplications followed by an addition. Thus, each evaluation of the sum in equation (2) requires N-1 computations and, since there are $N \cdot N^2 = N^3$ values of n and m for which the sum must be evaluated, the resulting number of computations is $(N-1)N^3$.) The computation method just proposed requires many fewer computations as will now be demonstrated.

The calculation of P_1 requires $2^{p-1} \, 2 \cdot 2^p = 2^{2p}$ computations and the calculation of P_ℓ, from $P_{\ell-1}$, $\ell = 2, \dots p-1$ requires $2^{p-\ell} 2^\ell 2^{p+\ell-1} = 2^{2p+\ell-1}$ computations. Finally, the calculation of P_p from P_{p-1} requires $2^p \, 2^{2p} = 2^{3p}$

computations. Thus, the total number of computations is given by

$$C = \sum_{\ell=1}^{p-1} 2^{2p+\ell-1} + 2^{3p} = \frac{1}{2} N^2(3N-2)$$

For large N, this figure is roughly a factor of $\frac{2}{3}$ N smaller than the number of computations required by straightforward evaluation of equation (2).

A further reduction in the number of computations can be effected if P need not be evaluated for all possible values of its arguments. For example, assume that P is to be evaluated for all velocity resolution cells but only for the M smallest acceleration cells where M is of the form $M = 2^{p+g}$, $0 \leq g < p$. (The reason for assuming M to be of this form will become apparent in a moment.) In this case, the binary expansion for m requires only p + g instead of 2p binary digits; i.e. $m = m_o + \ldots + m_{p+g-1} 2^{p+g-1}$. Examination of equations (4), (5), and (6) now reveals that the number of computations required for P_ℓ is equal to $2^{2p+\ell}$ for $\ell = 1, \ldots, g$ and equal to 2^{2p+g-1} for $\ell = g+1, \ldots p$. It follows that the total number of computations C_M is given by

$$C_M = \sum_{\ell=1}^{g} 2^{2p+\ell-1} + (p-g)\, 2^{2p+g-1}$$

$$= N(M-N) + \frac{NM}{2} \log_2 \left(\frac{N^2}{M} \right) \qquad (7)$$

It is interesting to compare the value of C_M given by equation (6) with the number of computations required by(two other methods) for evaluating P for N velocity resolutions cells and M acceleration resolution cells. Straightforward evaluation of equation (2) requires NM(N-1) computations; thus the efficiency of the above proposed method can be assessed by evaluating the ratio

$$\frac{C_M}{NM(N-1)} = \frac{1 - \frac{N}{M}}{N-1} + \frac{1}{2(N-1)} \log_2 \left(\frac{N^2}{M} \right) \qquad (8)$$

As a numerical example, consider the numbers N = M = 32 for which equation (8) yields $\frac{C_M}{NM(N-1)} = 0.08$. This illustrates the considerable computational advantage the proposed method has over straightforward evaluation of equation (2).

Another way of calculating P for N velocity resolution cells and M acceleration resolution cells is to combine the acceleration factor V^{mk^2} with the data r_k in equation (2) and then apply the Cooley-Tukey method for a pure velocity periodogram for each desired value of m. This approach results in a total of NM $\log_2 N$ computations which when compared with C_M yields

$$\frac{C_M}{NM \log_2 N} = \frac{(1 - \frac{N}{M}) + \frac{1}{2} \log_2 \left(\frac{N^2}{M} \right)}{\log_2 N} \qquad (9)$$

Substituting N = M = 32 in equation (8) results in $C_M/NM \log_2 N = 1/2$ which means that, in this case, our method is only a factor of two more efficient than the modified Cooley-Tukey method.

Distribution:

A. Bertolini
S. Catalano
D. Clark
D. DeLong
A. Galvin
A. Grometstein
H. Hoffman (3)
E. Hofstetter (5)
H. Jones
L. Martin
S. Miller
J. Morriello
J. Penhune
J. Perry
E. Schwarz
F. Schweppe
C. Slade
R. Straw
F. Vogt
R. Wishner

DOCUMENT CONTROL DATA – R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Lincoln Laboratory, M.I.T.	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP None

3. REPORT TITLE

An Efficient Technique for the Calculation of Velocity-Acceleration Periodograms

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Technical Note

5. AUTHOR(S) *(Last name, first name, initial)*

Hofstetter, Edward M.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
18 May 1966	14	1

8a. CONTRACT OR GRANT NO. AF 19(628)-5167 b. PROJECT NO. ARPA Order 498 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) Technical Note 1966-31 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)* ESD-TR-66-208

10. AVAILABILITY/LIMITATION NOTICES

Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES None	12. SPONSORING MILITARY ACTIVITY Advanced Research Projects Agency, Department of Defense

13. ABSTRACT

The Cooley-Tukey method for greatly reducing the number of computations required to evaluate a velocity periodogram has been extended to the evaluation of velocity-acceleration periodograms. For N data points, this method requires approximately a factor of 2/3 N fewer computations than would be required by straightforward evaluation of the periodogram.

14. KEY WORDS

periodograms